Introduction to control theory and applications

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Graduate course on Optimal control
University of Fukuoka
18/06/2015

What is control theory ?

- Control theory is a branch of mathematics that studies the properties of *control systems* i.e *dynamical systems whose behavior can be modified by a command*
- General Mathematical formalism of a control system :

$$\dot{x}(t) = f(t, x(t), u(t))$$

  where
  - $t \in [t_0 \, t_f]$ is the time variable,
  - x is the state variable defined on $[t_0 \, t_f]$ and valued in a smooth variable M,
  - u is a measurable bounded function defined on $[t_0 \, t_f]$, valued in a smooth variable U, called the control variable,
  - $f : \mathbf{R} \times M \times U \to TM$ is a smooth application.
- Goal : Bring the state variable from a given initial condition to a given final condition i.e solve a boundary value problem

$$\begin{cases} \dot{x}(t) = f(t, x(t), u(t)) \\ x(t_0) = x_0 \in M, \; x(t_f) = x_f \in M. \end{cases}$$

Schematic diagram

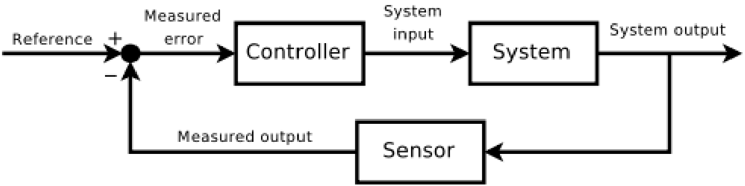


Figure: A dynamical system controlled by a *feedback loop*. We call *the error* the difference between *the reference* (the desired output) and the *the measured output*. This error is used by *the controller* to design a control on the system so that the measured output gets closer to the reference.

Source : Wikipedia

Questions that arise...

Controllability of a control system ? Is it possible to bring the state variable from any initial condition to any final condition in a finite time ?

- linear systems

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) + r(t) \\ x(t_0) = x_0 \end{cases}$$

  where $x(t)\ \mathbb{R}^n$, $u(t)\ \mathbb{R}^n$, $A(t) \in \mathcal{M}_n(\mathbb{R})$, $B(t) \in \mathcal{M}_{n,m}(\mathbb{R})$ and $r(t) \in \mathcal{M}_{n,1}(\mathbb{R})$ for all $t \in [t_0, t_f]$.

  $\rightarrow$ *Kalman condition*
  constraints on the control ?

- nonlinear systems

  $\rightarrow$ *way more difficult*
  $\rightarrow$ Poincaré's Reccurence theorem, Poisson-stability, linearization, local controllability

Stabilization of a control system ? How can we make a control system insensitive to perturbations ?

Example : if (x_e, u_e) is an equilibrium point of the autonomous control system

$$\dot{x}(t) = f(x(t), u(t))$$

i.e

$$f(x_e, u_e) = 0.$$

Does it exist a control u such that, for all $\epsilon > 0$, there exists $\eta > 0$ such that, for all $x_0 \in B(x_e, \eta)$ and all $t \geq 0$, the solution to the system

$$\begin{cases} \dot{x}(t) = f(t, x(t), u(t)) \\ x(t_0) = x_0 \end{cases}$$

satisfies $||x(t) - x_e|| \leq \epsilon$?

- linear systems $\rightarrow$ controlability
- nonlinear systems $\rightarrow$ Lyapunov functions

Optimal control? Can we determine the optimal solutions to a control system for a given optimization criterion ?

$\rightarrow$ Find the *solution* to the boundary value problem

$$\begin{cases} \dot{x}(t) = f(t, x(t), u(t)) \\ x(t_0) = x_0 \in M_0, \ x(t_f) = x_f \in M_1 \end{cases}$$

which *minimizes* the cost

$$\min_{u(.)\in U} \int_{t_0}^{t_f} f_0(t, x(t), u(t))dt + g(t_f, x_f)$$

where $f_0 : \mathbf{R} \times M \times U \rightarrow \mathbf{R}$ is smooth and $g : \mathbf{R} \times M \rightarrow \mathbf{R}$ is continuous.

$\int_{t_0}^{t_f} f_0(t, x(t), u(t))dt$: *Lagrange* cost

$g(t_f, x_f)$: *Mayer* cost

see

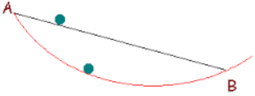
In 1696, *Johan Bernoulli* challenged his contemporary with the following problem :

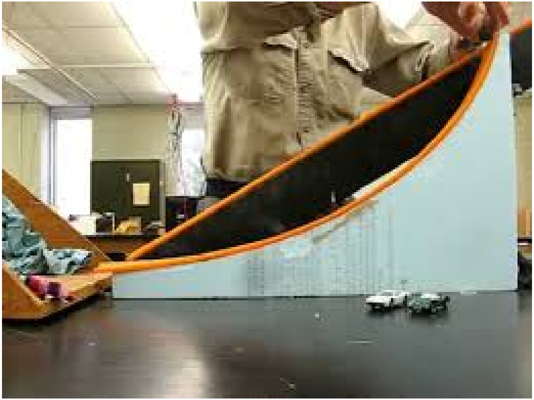
Consider *two points* A and B such that A is above B. Assume that a *object* is located a the point A with *no initial velocity* and is only subjected to the *gravity*. What is the curve between A and B so that the object travels from A to B in *minimal* time ?

Remark : We know that the *straight* line is the shortest way between two points. Is it the *fastest* way ?

NO !

The fastest way is a *cycloid arc* whose *tangent* line at the point A is *vertical*.





A short movie which illustrates this result.

Model of the *interaction* of *HIV* and *T-cells* in the immune system

$$\rightarrow \frac{dT(t)}{dt} = s_1 - \frac{s_2 V(t)}{B_1 + V(t)} - \mu T(t) - kV(t)T(t) + u_1(t)T(t)$$

$$\frac{dV(t)}{dt} = \frac{g(1-u_2(t))V(t)}{B_2 + V(t)} - cV(t)T(t).$$

where

- $T(t)$: Concentration of unaffected T cells
- $V(t)$: Concentration of HIV particles
- (u_1, u_2) : Control terms, action of the treatment
- $s_1 - \frac{s_2 V(t)}{B_1 + V(t)}$: proliferation of unaffected T cells
- $\mu T(t)$: natural loss of unaffected T cells
- $kV(t)T(t)$: loss by infection
- $\frac{g(1-u_2(t))V(t)}{B_2 + V(t)}$: proliferation of virus
- $cV(t)T(t)$: viral loss

Objective : *maximizing* the *efficiency* of the treatment

$$\rightarrow \max_{(u_1,u_2)} \int_0^{t_f} T(t) - (A_1 u_1^2(t) + A_2 u_2^2(t))dt.$$

i.e

- maximizing the *number of unaffected T cells* during the treatment
- *and* minimizing the *systemic cost* of the treatment

A_1, A_2 : 2 constants/weights
$A_1 u_1^2(t) + A_2 u_2^2(t)$: severity of side effects of the treatment.

Results : The optimal control (u_1, u_2) can be written as a *feeback* control i.e function of T and V. The *optimal synthesis* can be simulated *numerically*.

Context : An *economy* consisting of *2 sectors*. The sector 1 produces financial goods and the sector 2 produces consumption goods. Denote $x_1(t)$ and $x_2(t)$ the *productions* in sectors 1 and 2 and $u(t)$ the *proportion of investment* allocated to sector 1.

Assumption : Increase in production in each sector is *proportional* to the investment allocated to each sector.

Problem : *Maximizing* the *total consumption* over interval of time $[0, T]$.

$\rightarrow$ Optimal control problem

$$\begin{cases} \dot{x}_1(t) = \alpha u(t) x_1(t) \\ \dot{x}_2(t) = \alpha(1 - u(t)) x_2(t) \\ \max_{u(.) \in U} \int_0^T x_2(t) dt \\ x_1(0) = a_1, \; x_2(0) = a_2 \end{cases}$$

where α is some constant of proportionality and a_1 and a_2 are the initial productions in sectors 1 and 2.

This problem can be solved *analytically*. The optimal solution (u^*, x_1^*, u_2^*) is

$$u^*(t) = \begin{cases} 1 \text{ if } 0 \leq t \leq T - \frac{\alpha}{2} \\ 0 \text{ if } T - \frac{\alpha}{2} < t \leq T \end{cases}$$

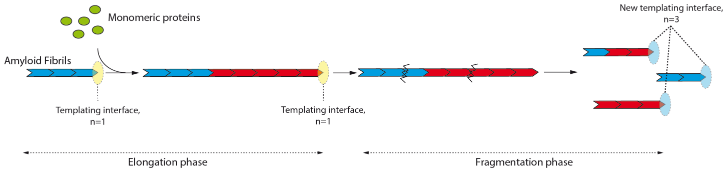
$$x_1^*(t) = \begin{cases} a_1 e^{\alpha t} \text{ if } 0 \leq t \leq T - \frac{\alpha}{2} \\ a_1 e^{\alpha T - 2} \text{ if } T - \frac{\alpha}{2} < t \leq T \end{cases}$$

$$x_2^*(t) = \begin{cases} a_2 \text{ if } 0 \leq t \leq T - \frac{\alpha}{2} \\ a_2 e^{(\alpha t - \alpha T + 2) e^{\alpha T - 2}} \text{ if } T - \frac{\alpha}{2} < t \leq T \end{cases}$$

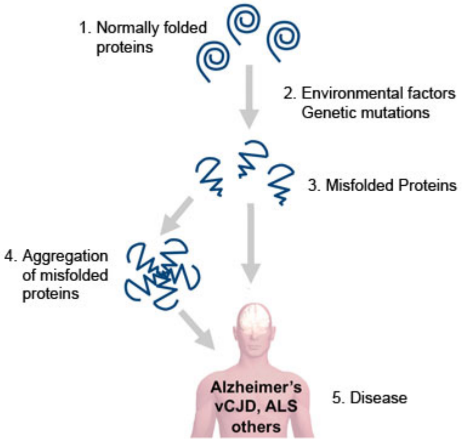
Identification of the Fragmentation Role in the Amyloid Assembling Processes and Application to their Optimization

Context : Apply techniques from geometric control to a kinetic model of amyloid formation which will take into account the contribution of fragmentation to the de novo creation of templating interfaces to design optimal strategies for accelerating the current amplification protocols, such as the Protein Misfolding Cyclic Amplification (PMCA). The objective is to reduce the time needed to diagnose many neurodegenerative diseases.

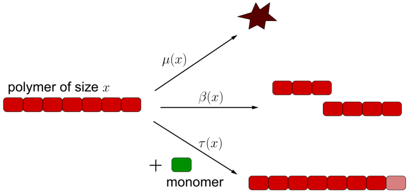
Fibril fragmentation : Fibril fragmentation has been reported to enhance the polymerization process underlying the behavior of some specific prions. There is a significant lack of knowledge concerning the fragmentation process and the de novo generation of templating interfaces, both in the mechanisms of its occurrence and its contribution to the acceleration of the pathology.



- *Transmissible spongiform encephalopathies (TSEs)*



- Aggregation-Fragmentation



We denote by $x_i^l(t)$, $l = 1, \cdots, k_i$, the density of polymers of size i in compartment l at a given time t. The corresponding rate of change due to elongation is then described as follows :

$$r(u(t))\Big[\sum_{s=1}^{k_{i-1}} \tau_{i-1}^{l,s} x_{i-1}^s - \sum_{r=1}^{k_{i+1}} \tau_i^{r,l} x_i^l\Big].$$

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The parameter $\tau_{i-1}^{l,s}$ is the growth rate of polymers of size $i-1$ in compartment s that grow in compartment l of polymers of size i, and $\tau_i^{r,l}$ is the growth rate of polymers of size i in compartment l that grow into polymers of size $i+1$ (in compartment r).

Compartmental model of amyloid formation

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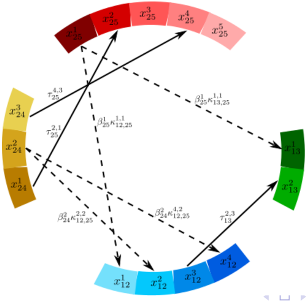
Compartmental model of amyloid formation

The fragmentation rate of change is expressed by the fact that polymers of a given size and given compartment fragment into polymers of a given size and compartment at different rates :

$$u(t)\left[2\sum_{j=i+1}^{n}\sum_{s=1}^{k_j}\beta_j^s\kappa_{ij}^{l,s}x_j^s - \beta_i^l x_i^l\right]$$

where β_i^l (β_i^s) represents the fragmentation coefficient of polymer of size i in compartment l (s) and the coefficient $\kappa_{ij}^{l,s}$ captures the fraction of polymer of size j that fragment from compartment s into size i polymer in compartment l.



Model

To summarize, we propose a model of the form :

$$\dot{x}_i^l(t) = r(u(t))\Big[\sum_{s=1}^{k_{i-1}} \tau_{i-1}^{l,s} x_{i-1}^s(t) - \sum_{r=1}^{k_{i+1}} \tau_i^{r,l} x_i^l(t)\Big] +$$

$$u(t)\Big[2\sum_{j=i+1}^{n}\sum_{s=1}^{k_j} \beta_j^s \kappa_{ij}^{l,s} x_j^s(t) - \beta_i^l x_i^l(t)\Big]$$

It can be written in a matrix form :

$$\dot{x}(t) = \big(u(t)A + r(u(t))B\big)x(t).$$

The growth matrix, B, and the fragmentation matrix, A, as well as vector $x(t)$, have a block structure with blocks corresponding to the different compartments. The parameters will be determined experimentally. The behavior inside each compartment also needs to be determined. In particular the in vitro elongation of the fibrils appears to saturate after some time and the polymerization process is then blocked. This saturation effect has to be understood and included in the model $\rightarrow$ *nonlinear*, which raises new challenging mathematical questions.

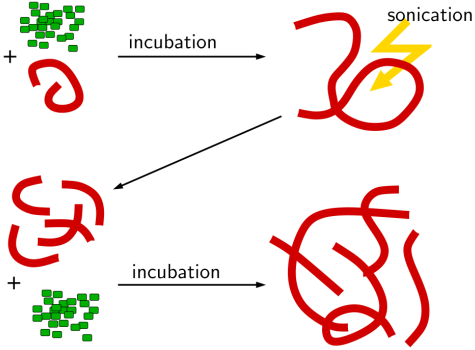
Optimization of amplification protocols

- Since incubation of a disease triggered by prions can take place over very long period of time, an important question is *the optimization of the templating, elongation, and polymerization processes* to accelerate the detection of the protein in an affected person : PMCA.

- Typically, during the PMCA the incubation phase (no sonication) is more than 30 times the duration of the sonication phase (at a constant frequency) and alteration of these two phases takes place over 48 hours. This correspond to a bang-bang strategy with the control (sonication intensity) switching a finite number of times between its minimum and maximum values.

Protein Misfolding Cyclic Amplification

- protocol to amplify the quantity of aggregates



The general expression for our system is of the form :

$$\dot{x}(t) = [Au(t) + B(t, x(t))r(u(t))]x(t),$$
$$x(0) = x_0 > 0,$$

where $x = (x_1^1, \cdots, x_1^{k_1}, \cdots, x_n^1, \cdots, x_n^{k_n}) \in R^m,\ m = \sum_{i=1}^n k_i$.

The matrix A is constant since we assume that the fragmentation coefficients stay constant throughout the protocol. However, the elongation coefficients might vary with time to reflect the saturation hypothesis. This implies that the matrix B is not constant but can depend explicitly on t or on the current density of polymers $x(t)$.

Optimal Cost : final density of polymers, $c(x(T)) = \sum_{i=1}^n (i \sum_{j=1}^{k_i} x_i^j(T))$.

We make the assumption that the function r is a decreasing convex function. This will be checked experimentally, and adapted if it is necessary in further work. Using a reparametrization and some assumptions on r, we can rewrite the optimal problem as an affine single-input system :

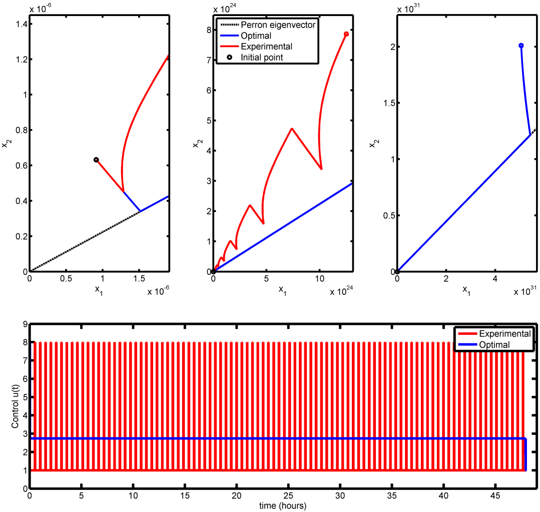
$$\dot{x}(t) = f_0(t, x(t)) + f_1(t, x(t))u(t), \quad (1)$$

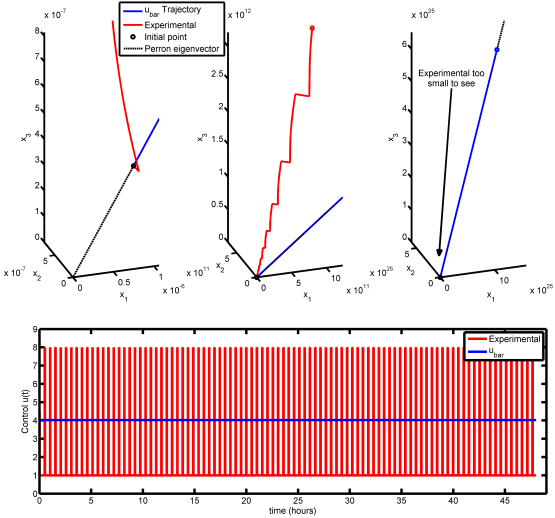
$$x(0) = x_0 > 0, \quad (2)$$

$$\min_{u_{\min} \leq u \leq u_{\max}} -\psi x(T), \quad (3)$$

where $f_0(t, x(t)) = B(t, x(t))$ and $f_1(t, x(t)) = Ax(t) + aB(t, x(t))$, $a < 0$. Our optimal control problem is in *Mayer form* with fixed time T but not constraints on the terminal state $x(T)$.

2D Simulations




u_bar Trajectory
Experimental
Initial point
Perron eigenvector
Experimental too small to see
x 10^-7
x 10^12
x 10^25
x_3
x_2
x_1
Experimental
u_bar
Control u(t)
time (hours)

Optimal control theory is at the *crossroad* of

- Theory of Differential Equations/Dynamical Systems (Finding solutions to *differential systems*, problem of *existence* and *unicity* of optimal solutions)
- Differential geometry (optimal synthesis strongly depends on the *geometric properties* of the problem, *modern theory* of optimal control)
- Optimization
- Modeling (*relevance* of the way that an optimal control problem is *set up*)
- Numerical Analysis (*numerical methods* to approximate optimal solutions)
- Applications (solving *real world* problems)

Objective : use optimal control theory to compute *optimal space transfers* in the *Earth-Moon* system

- *time*-minimal space transfers
- *energy*-minimal space transfers

First question : How to *model* the *motion* of a spacecraft in the Earth-Moon system ?

- *Neglect the influences of other planets*
- The spacecraft *does not* affect the motion of the Earth and the Moon
- Eccentricity of orbit of the Moon is *very small* (≈ 0.05)

The motion of the spacecraft in the Earth-Moon system can be modelled by *the planar restricted 3 body problem*.

Description :

- The Earth (mass M_1) and Moon (mass M_2) are *circularly* revolving aroud their *center of mass* G.
- The spacecraft is *negligeable point mass* M involves in the plane defined by the Earth and the Moon.
- Normalization of the masses : $M_1 + M_2 = 1$
- Normalization of the distance :$d(M_1, M_2) = 1$.

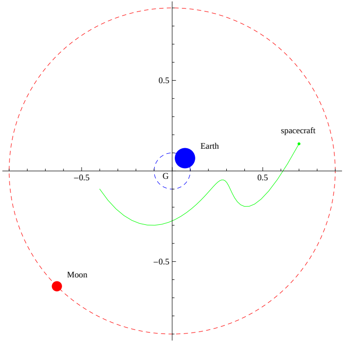


Figure : The circular restricted 3-body problem. The blue dashed line is the orbit of the Earth and the red one is the orbit of the Moon. The trajectory of spacecraft lies in the plan deined by these two orbits.

Idea : Instead of considering a fixed frame $\{G, X, Y\}$, we consider a *dynamic rotating* frame $\{G, x, y\}$ which rotates with the same angular velocity as the Earth and the Moon.

→ *rotation* of angle t

→ *substitution*

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos(t)x + \sin(t)y \\ -sin(t)x + \cos(t)y \end{pmatrix}$$

→ *simplifies* the equations of the model

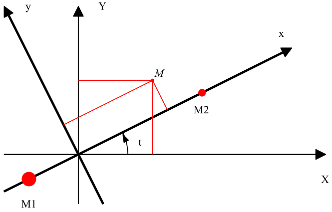


Figure : Comparision between the fixed frame $\{G, X, Y\}$ and the rotating frame $\{G, x, y\}$.

In the rotating frame

- define the *mass ratio* $\mu = \frac{M_2}{M_1+M_2}$
- the *Earth* has mass $1-\mu$ and is located at $(-\mu, 0)$;
- the *Moon* has mass μ and is located at $(1-\mu, 0)$;
- *Equations of motion*

$$\begin{cases} \ddot{x} - 2\dot{y} - x = \frac{\partial V}{\partial x} \\ \ddot{y} + 2\dot{x} - y = \frac{\partial V}{\partial y} \end{cases}$$

where

$-V$: is the *mechanical potential*

$$V = \frac{1-\mu}{\varrho_1^3} + \frac{\mu}{\varrho_2^3}$$

ϱ_1 : *distance* between the *spacecraft* and the *Earth*

$$\varrho_1 = \sqrt{(x+\mu)^2 + y^2}$$

ϱ_2 :*distance* between the *spacecraft* and the *Moon*

$$\varrho_2 = \sqrt{(x-1+\mu)^2 + y^2}.$$

Legendre transformation

$$(q_1, q_2) = (x, y), \;\; p = (p_1, p_2) = (\dot{q}_1 - q_2, \dot{q}_1 + q_2)$$

$\rightarrow$ Equations of motion becomes an *Hamiltonian* system

$$\dot{q} = \frac{\partial H}{\partial p}(q(t), p(t)), \;\; \dot{p} = -\frac{\partial H}{\partial q}(q(t), p(t))$$

where

$$H(q, p) = \frac{1}{2}\|p\|^2 + p_1 q_2 - p_2 q_1 - \frac{1-\mu}{\varrho_1} - \frac{\mu}{\varrho_2} + \frac{\mu(1-\mu)}{2}.$$

Remark :

$$\frac{dH}{dt} = \frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial p}\dot{p} = \frac{\partial H}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial H}{\partial q} = 0$$

$\rightarrow$ The value of H is *constant* along a *trajectory* of the planar restricted 3-body problem

$\rightarrow$ H is a *first integral* of the problem

Hill Regions

There are 5 *possible* regions of motion, know as the *Hill regions*

Each region is defined by the value of the Hamiltonian H (*total energy* of the system)

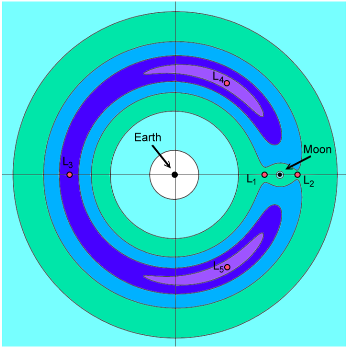


Figure: The Hill regions of the planar restricted 3-body problem

Toplogy/Shape of the regions is determined with respect to the total energy at the *equilibrium points* of the system

Equilibrium points

Critical points of the mechanical potential

→ Points (x, y) where $\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} = 0$

- *Euler points* : colinear points L_1,L_2, L_3 located on the axis $y = 0$, with

$$x_1 \simeq 1.1557, \quad x_2 \simeq 0.8369, \quad x_1 \simeq -1.0051.$$

- *Lagrange points* : L_4, L_5 which form equilateral triangles with the primaries.

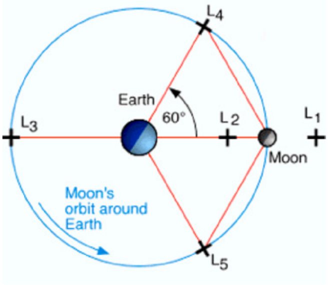


Figure: Equilibrium points of the planar restricted 3-body problem

Control on the motion of the spacecraft ?
→ Thrust/Propulsion provided by the engines of the spacecraft
→ control term $u = (u_1, u_2)$ must be added to the equations of motion
→ *controlled dynamics* of the spacecraft

$$\begin{cases} \ddot{x} - 2\dot{y} - x = \frac{\partial V}{\partial x} + u_1 \\ \ddot{y} + 2\dot{x} - y = \frac{\partial V}{\partial y} + u_2 . \end{cases}$$

Setting $q = (x, y, \dot{x}, \dot{y})$
→ *bi-input system*

$$\dot{q} = F_0(q) + F_1(q)u_1 + F_2(q)u_2$$

where

$$F_0(q) = \begin{pmatrix} q_3 \\ q_4 \\ 2q_4 + q_1 - (1-\mu)\frac{q_1+\mu}{((q_1+\mu)^2+q_2^2)^{\frac{3}{2}}} - \mu\frac{q_1-1+\mu}{((q_1-1+\mu)^2+q_2^2)^{\frac{3}{2}}} \\ -2q_3 + q_2 - (1-\mu)\frac{q_2}{((q_1+\mu)^2+q_2^2)^{\frac{3}{2}}} - \mu\frac{q_2}{((q_1-1+\mu)^2+q_2^2)^{\frac{3}{2}}} \end{pmatrix},$$

$$F_1(q) = \frac{\partial}{\partial q_3},, \; F_2(q) = \frac{\partial}{\partial q_4}$$